Surfaces

Level Surfaces

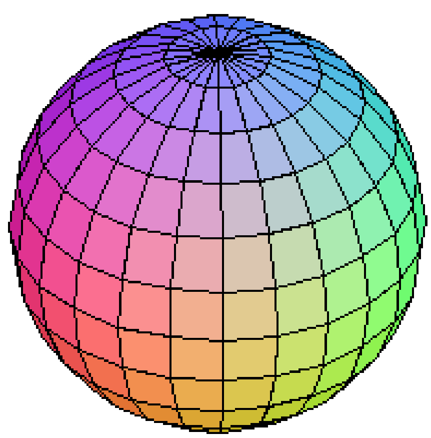
One of the goals of this chapter is to use differential calculus to explore *surfaces*, in much the same way that we used differential calculus to study curves in the first chapter. In this section, we introduce two types of surfaces and some of their properties.

To begin with, the *level surface* of level k of a function $U(x, y, z)$ is defined to be the set of all points in R^3 which are solutions to

$$U(x, y, z) = k$$

Indeed, many of the most familiar surfaces are level surfaces of functions of 3 variables.

> *EXAMPLE 1* Find the equation of a sphere of radius R centered at the origin.
>
> **Solution:** Every point (x, y, z) on the sphere must be a distance R from the origin. Thus, the length of every vector with initial point $(0, 0, 0)$ and final point (x, y, z) is R, which means that
>
> $$\sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} = R$$
>
> This in turn simplifies to the following
>
> $$x^2 + y^2 + z^2 = R^2,$$

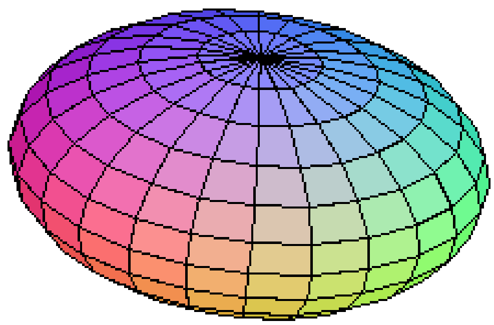


A *quadric surface* is a level surface of a second degree polynomial $Q(x, y)$. Indeed, the sphere of radius R centered at the origin is a level surface of level $k = R^2$ of the second degree polynomial

$$Q(x, y) = x^2 + y^2 + z^2$$

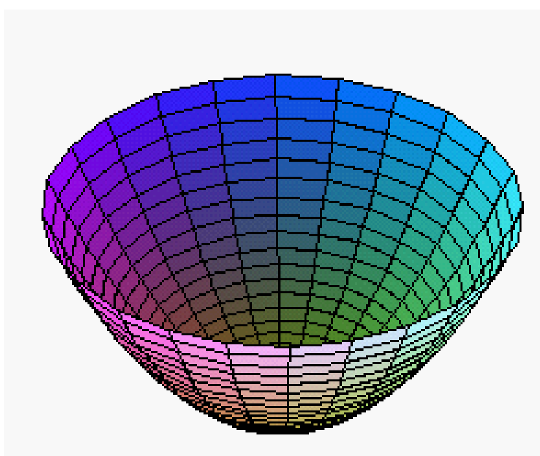
For example, an *ellipsoid* is a surface of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \ ,$$



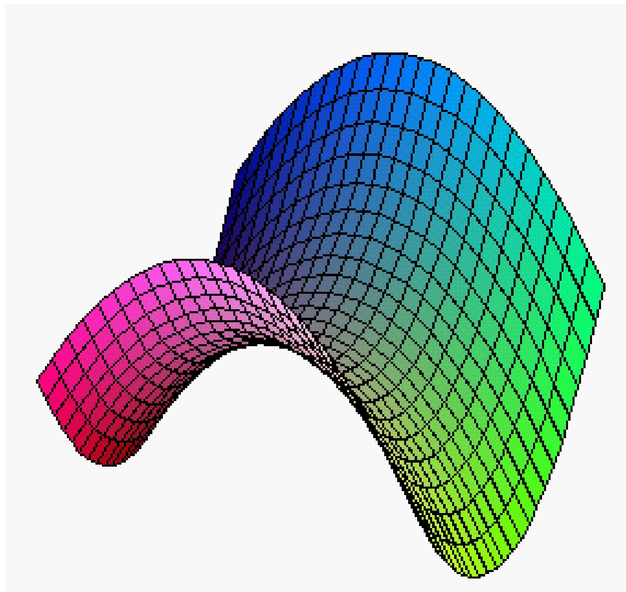
Other quadric surfaces include the *elliptic paraboloids,* which are defined by equations of the form

$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2},$$



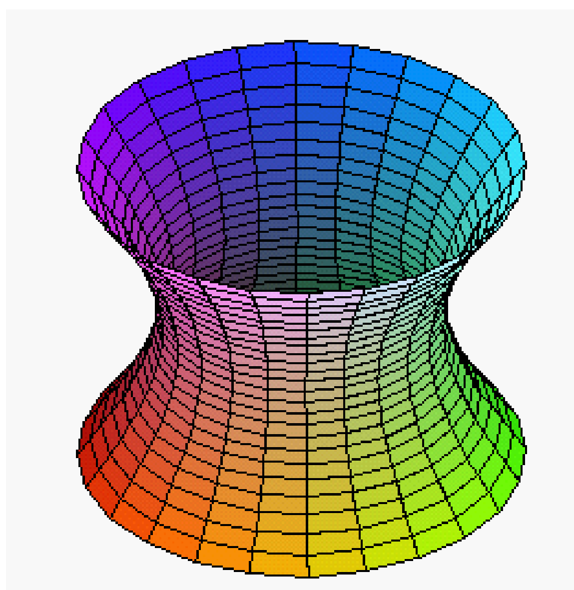
and the *hyperbolic paraboloids,* which are defined by equations of the form

$$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2},$$



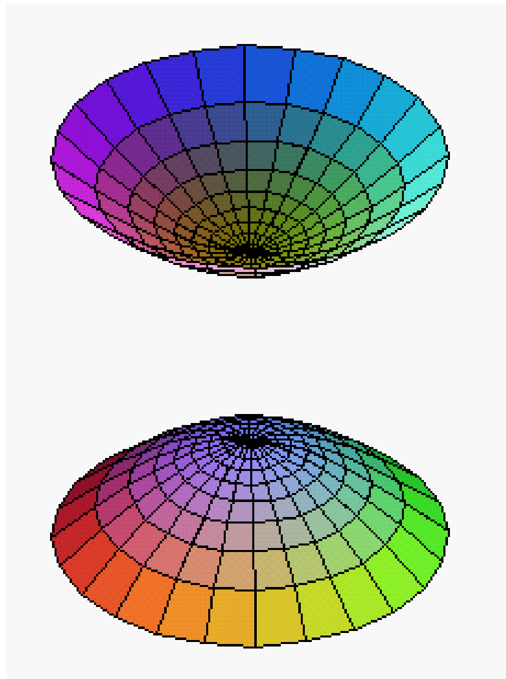
In addition, there are the *hyperboloids,* where a *hyperboloid in one sheet* has an equation of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$$



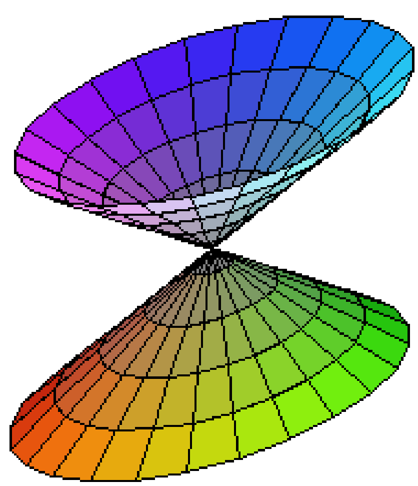
and a *hyperboloid in 2 sheets* has an equation of the form

$$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$



And finally, one of the most important classes of quadric surfaces are the *elliptic cones,* which are surfaces defined by equations of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2},$$



For example, the surface defined by $x^2 + y^2 = z^2$ is a right cylindrical cone, of the type used to define the conics (see the end of this section):

Check your Reading: What type of quadric surface is given by the equation

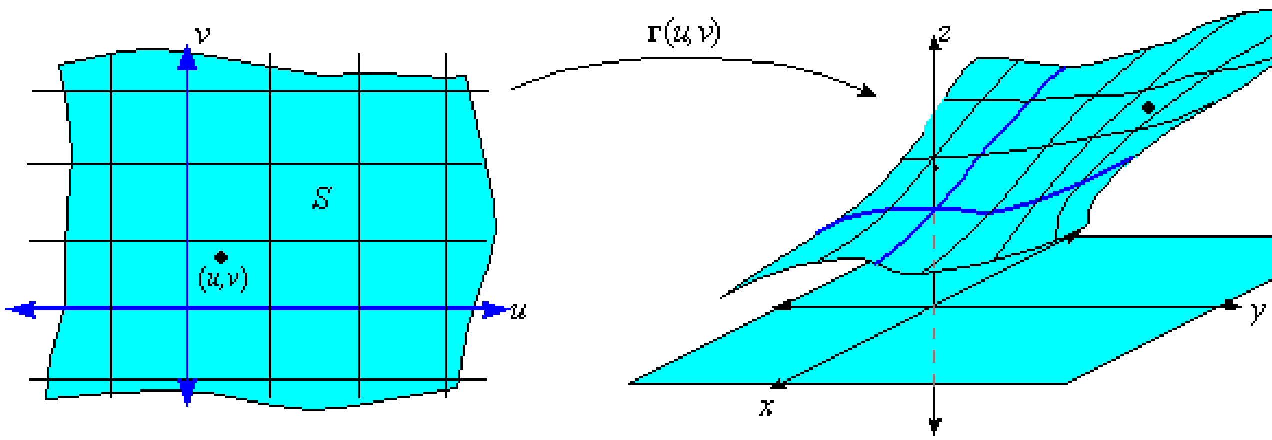
$$x^2 - y^2 + z^2 = 4$$

Parametric Surfaces

Surfaces can also be defined *parametrically.* In particular, suppose each component of a vector-valued function is a function of two variables u and v:

$$\mathbf{r}(u,v) = \langle f(u,v), g(u,v), h(u,v) \rangle \tag{1}$$

Then the graph of $\mathbf{r}(u,v)$ over some region R in the uv-plane is a surface in $\mathbb{R}^3$, and $\mathbf{r}(u,v)$ is called a *parametrization* of that surface.



Equivalently, $x = f(u,v)$, $y = g(u,v)$, and $z = h(u,v)$ for u, v in R defines a surface, and the variables u and v are often called the *coordinates* of the surface.

Given a parametric surface (1), we often desire to transform it into a level surface representation of the form $U(x,y,z) = k$. Finding a level surface representation often requires the use of trigonometric identities, such as

$\cos^2(t) + \sin^2(t) = 1$	$1 + \tan^2(t) = \sec^2(t)$	$1 - 2\sin^2(t) = \cos(2t)$
$\cosh^2(t) - \sinh^2(t) = 1$	$1 + \cot^2(t) = \csc^2(t)$	$2\cos^2(t) - 1 = \cos(2t)$

Other identities that may be necessary include $2\sin(t)\cos(t) = \sin(2t)$ and $e^t e^{-t} = 1$.

EXAMPLE 2 Find a level surface representation of the surface parameterized by

$$\mathbf{r}(u,v) = \langle \cos(u)\cosh(v), \sin(u)\cosh(v), \sinh(v) \rangle$$

Solution: Since $x = \cos(u)\cosh(v)$, $y = \sin(u)\cosh(v)$, and $z = \sinh(v)$, the identity $\cos^2(u) + \sin^2(u) = 1$ leads to

$$\begin{aligned} x^2 + y^2 &= \cos^2(u)\cosh^2(v) + \sin^2(u)\cosh^2(v) \\ &= \cosh^2(v)\left[\cos^2(u) + \sin^2(u)\right] \\ &= \cosh^2(v) \end{aligned}$$

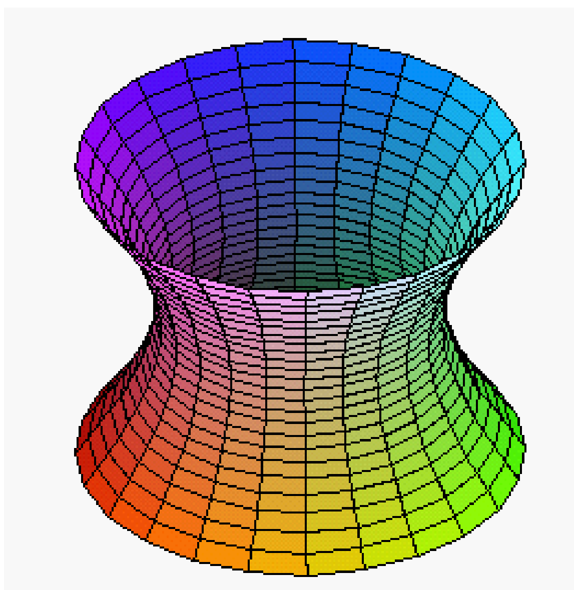
As a result, the identity $\cosh^2(v) - \sinh^2(v) = 1$ leads to

$$x^2 + y^2 - z^2 = \cosh^2(v) - \sinh^2(v) = 1$$

Thus, $\mathbf{r}(u,v) = \langle \cos(u)\cosh(v), \sin(u)\cosh(v), \sinh(v) \rangle$ is a parametrization of the level surface

$$x^2 + y^2 - z^2 = 1$$

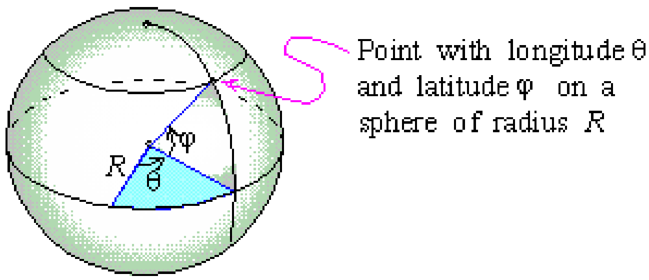
which we recognize as a hyperboloid in one sheet.



A sphere of radius R centered at the origin is often parametrized in terms of *longitude* θ and *latitude* φ, which results in the parameterization

$$\mathbf{r}(\theta,\varphi) = \langle R\cos(\varphi)\cos(\theta), R\cos(\varphi)\sin(\theta), R\sin(\varphi) \rangle \tag{2}$$

where θ in $[0, 2\pi]$ and φ is in $[-\pi/2, \pi/2]$.



EXAMPLE 3 Show that (2) is a parameterization of the sphere of radius R centered at the origin.

Solution: Since $x = R\cos(\varphi)\cos(\theta)$, $y = R\cos(\varphi)\sin(\theta)$ and $z = R\sin(\varphi)$, the identity $\cos^2(\theta) + \sin^2(\theta) = 1$ leads to

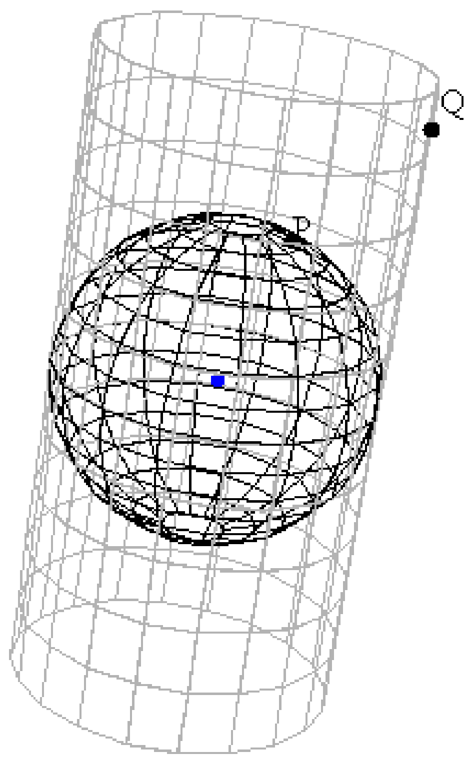
$$\begin{aligned} x^2 + y^2 &= R^2\cos^2(\varphi)\cos^2(\theta) + R^2\cos^2(\varphi)\sin^2(\theta) \\ &= R^2\cos^2(\varphi)\left[\cos^2(\theta) + \sin^2(\theta)\right] \\ &= R^2\cos^2(\varphi) \end{aligned}$$

Moreover, $z^2 = R^2\sin^2(\varphi)$ implies that

$$x^2 + y^2 + z^2 = R^2\cos^2(\varphi) + R^2\sin^2(\varphi) = R^2$$

which is the equation of the sphere of radius R centered at the origin.

However, cartographers and mathematicians have long used parameterizations of the sphere other than (2). For example, in 1599, the mapmaker Gerard Mercator constructed a projection of the earth's surface in which a straight line on a map corresponds to a fixed compass bearing on the earth's surface. To do so, he first imagined that the sphere was inside of a cylinder with radius R.



This led to the *Mercator parametrization* of the sphere:

$$\mathbf{r}(\theta, \mu) = \langle R\,\mathrm{sech}(\mu)\cos(\theta), R\,\mathrm{sech}(\mu)\sin(\theta), R\tanh(\mu)\rangle$$

where the hyperbolic secant and tangent functions are defined

$$\mathrm{sech}(\mu) = \frac{1}{\cosh(\mu)}, \qquad \tanh(\mu) = \frac{\sinh(\mu)}{\cosh(\mu)}$$

We will examine the Mercator parametrization more closely in the exercises.

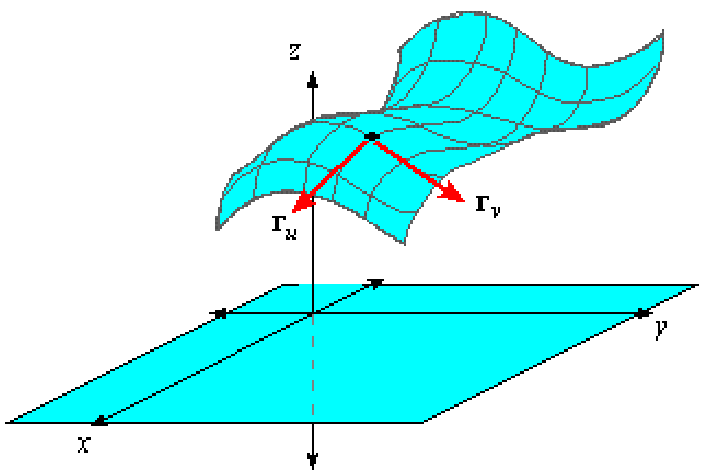
Check your Reading: How would you define $\mathrm{csch}(t)$?

Tangent Vectors to Parametric Surfaces

If q is a constant, then $\mathbf{r}(u, q)$ is a function of u only and is therefore a curve (specifically, it is the v-curve for $v = q$). As a result, the velocity vector

$$\mathbf{r}_u = \langle f_u, g_u, h_u \rangle$$

is tangent to the curve. Since the curve lies on the surface parametrized by $\mathbf{r}(u,v) = \langle f(u,v), g(u,v), h(u,v)\rangle$, it follows that $\mathbf{r}_u(p,q)$ is tangent to the surface at the point with coordinates (p,q). Likewise, we define $\mathbf{r}_v = \langle f_v, g_v, h_v\rangle$ and similarly it follows that $\mathbf{r}_v(p,q)$ is also tangent to the surface at the point corresponding to (p,q).



Moreover, we say that the parametrization is *regular* at a point P on the surface if $\mathbf{r}_u$ and $\mathbf{r}_v$ are nonzero and non-parallel at P; otherwise, the parametrization is said to be *singular* at P.

EXAMPLE 4 Find $\mathbf{r}_u$ and $\mathbf{r}_v$ for the sphere

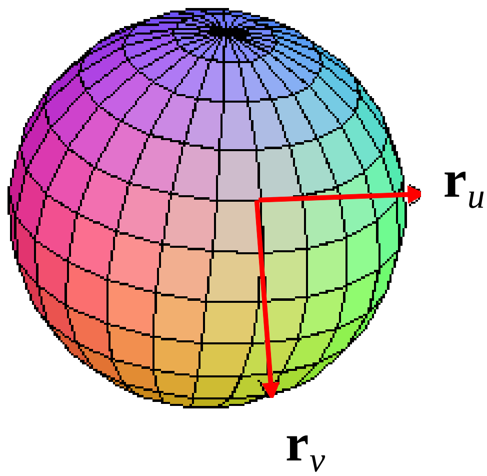
$$\mathbf{r}(u,v) = \langle \cos(u)\sin(v), \sin(u)\sin(v), \cos(v)\rangle$$

Solution: To find $\mathbf{r}_u$, we apply ∂_u to $\mathbf{r}$ to obtain

$$\mathbf{r}_u = \langle -\sin(u)\sin(v), \cos(u)\sin(v), 0\rangle$$

Likewise, to find $\mathbf{r}_v$, we apply ∂_v to $\mathbf{r}$ to obtain

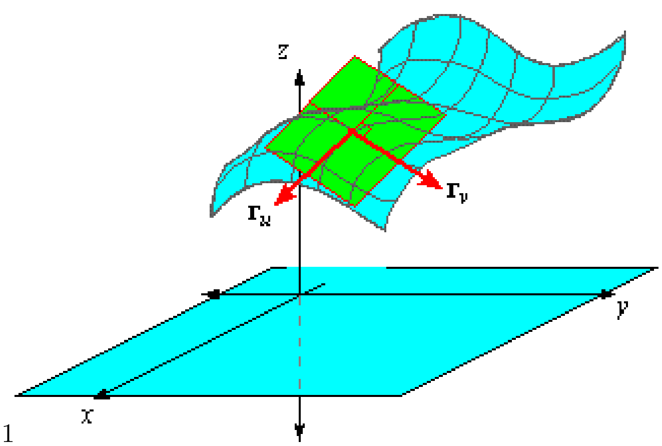
$$\mathbf{r}_v = \langle \cos(u)\cos(v), \sin(u)\cos(v), -\sin(v)\rangle$$



Notice in example 3 that

$$\mathbf{r}_u \cdot \mathbf{r}_v = -\sin(u)\sin(v)\cos(u)\cos(v) + \cos(u)\sin(v)\sin(u)\cos(v) = 0$$

That is, the vectors $\mathbf{r}_u$ and $\mathbf{r}_v$ are perpendicular for all (u, v). Parameterizations in which $\mathbf{r}_u$ and $\mathbf{r}_v$ are orthogonal for all (u, v) are important in applications because they introduce an *orthogonal* coordinate system in each tangent plane:



1

If $\mathbf{r}_u \cdot \mathbf{r}_v = 0$ for all (u, v), then we say that the parametrization $\mathbf{r}(u, v)$ is *orthogonal.*

EXAMPLE 5 Find $\mathbf{r}_u$ and $\mathbf{r}_v$ and determine if the following parametrization is orthogonal:

$$\mathbf{r} = \left\langle u^2 - v^2, 2uv, u^2 + v^2 \right\rangle$$

Solution: To find $\mathbf{r}_u$, we apply ∂_u to $\mathbf{r}$ to obtain

$$\mathbf{r}_u = \langle 2u, 2v, 2u \rangle$$

Likewise, to find $\mathbf{r}_v$, we apply ∂_v to $\mathbf{r}$ to obtain

$$\mathbf{r}_v = \langle 2v, 2u, -2v \rangle$$

The dot product of $\mathbf{r}_u$ with $\mathbf{r}_v$ is given by

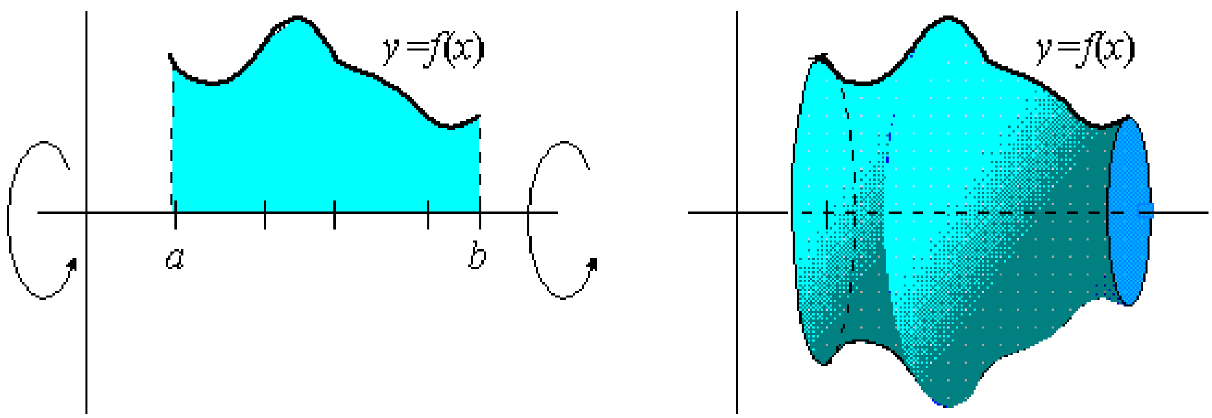
$$\mathbf{r}_u \cdot \mathbf{r}_v = 4uv + 4uv - 4uv = 4uv$$

Since $\mathbf{r}_u \cdot \mathbf{r}_v \neq 0$, the parametrization is **not** orthogonal.

Check your Reading: Are all orthogonal parameterizations also regular at every point? Explain.

Surfaces of Revolution

If $f(x) \geq 0$ for all x in $[a, b]$, then the revolution of $y = f(x)$ about the x-axis results in a *surface of revolution.*

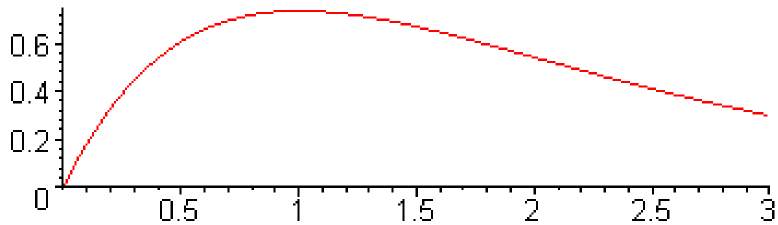


As a parametric surface, this surface of revolution can be represented by

$$\mathbf{r}(u, v) = \langle u, f(u)\sin(v), f(u)\cos(v) \rangle$$

where (u, v) is in $[a, b] \times [0, 2\pi]$. In single variable calculus, we derived formulas for calculating the volume of a solid bounded by a surface of revolution.

EXAMPLE 6 Find the equation of the surface of revolution obtained by revolving the curve $y = xe^{-x}$ for x in $[0, 3]$ about the x-axis

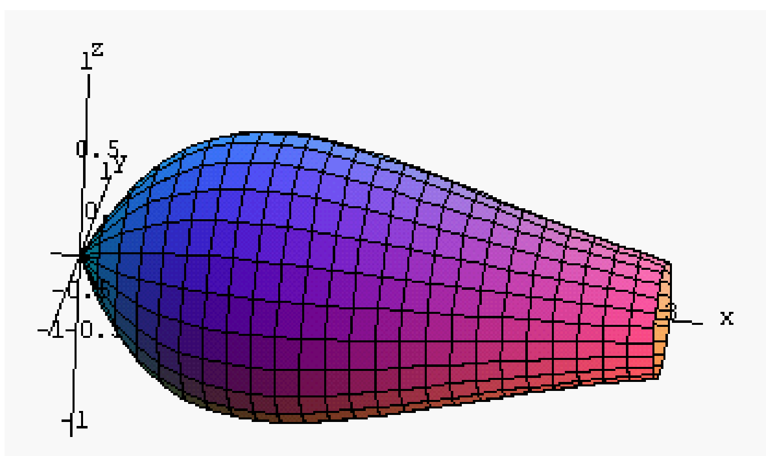


Find the tangent vectors $\mathbf{r}_u$ and $\mathbf{r}_v$. Is the parametrization orthogonal?

Solution: To do so, we notice that $f(u) = ue^{-u}$, so that the parametrization is

$$\mathbf{r}(u, v) = \left\langle u, ue^{-u}\sin(v), ue^{-u}\cos(v)\right\rangle$$

The graph of $\mathbf{r}(u, v)$ is shown below for (u, v) in $[0, 3] \times [0, 2\pi]$.



Notice now that $\mathbf{r}_u = \langle 1, (e^{-u} - ue^{-u})\sin(v), (e^{-u} - ue^{-u})\cos(v)\rangle$ and

$$\mathbf{r}_v = \left\langle 0, -ue^{-u}\cos(v), ue^{-u}\sin(v)\right\rangle$$

Moreover, it is rather easy to show that $\mathbf{r}_u \cdot \mathbf{r}_v = 0$, which implies that the parametrization is orthogonal.

More generally, if $f(u) \geq 0$ for all u in $[a, b]$, then

$$\mathbf{r}(u, v) = \langle g(u), f(u)\sin(v), f(u)\cos(v)\rangle$$

is the surface obtained by revolving the curve

$$\mathbf{r}(u, 0) = \langle g(u), 0, f(u)\rangle$$

about the x-axis, where u is in $[a, b]$ and v is in $[0, 2\pi]$.

EXAMPLE 7 What surface of revolution is obtained from revolving

$$\mathbf{r}(u, 0) = \langle \cos(u), 0, 3 + \sin(u)\rangle$$

about the x-axis? Find the tangent vectors $\mathbf{r}_u$ and $\mathbf{r}_v$. Is the parametrization orthogonal?

Solution: Since $\mathbf{r}(u, 0)$, u in $[0, 2\pi]$, is a circle of radius 1 centered at $(0, 0, 3)$, the surface is a torus parameterized by

$$\mathbf{r}(u, v) = \langle \cos(u), (3 + \sin(u))\sin(v), (3 + \sin(u))\cos(v)\rangle$$

The graph of $\mathbf{r}(u, v)$ is shown below for (u, v) in $[0, 2\pi] \times [0, 2\pi]$.

$$javaviewtorus$$

Notice now that $\mathbf{r}_u = \langle -\sin(u), \cos(u)\sin(v), \cos(u)\cos(v)\rangle$ and

$$\mathbf{r}_v = \langle 0, -(3 + \sin(u))\cos(v), (3 + \sin(u))\sin(v)\rangle$$

It follows that $\mathbf{r}_u \cdot \mathbf{r}_v = 0$, which implies that the parametrization is orthogonal.

Exercises

Use a graphing calculator or computer algebra system to sketch each surface and then find the level surface representations of each of the following parametric equations. Also, calculate $\mathbf{r}_u$ and $\mathbf{r}_v$ and determine if the parameterization is orthogonal.

1. $\mathbf{r} = \langle v\sin(u), v\cos(u), v\rangle$
2. $\mathbf{r} = \langle v\sin(u), v, v\cos(u)\rangle$
3. $\mathbf{r} = \langle \sin(u)\cos(v), \cos(u), \sin(u)\sin(v)\rangle$
4. $\mathbf{r} = \langle \sin(v)\sin(u), \cos(v)\sin(u), \cos(u)\rangle$
5. $\mathbf{r} = \langle 2\sin(u)\cos(v), 3\cos(u), 2\sin(u)\sin(v)\rangle$
6. $\mathbf{r} = \langle 2\sin(u)\cos(v), 2\cos(u), \sin(u)\sin(v)\rangle$
7. $\mathbf{r} = \langle \sin(u)\cosh(v), \sinh(v), \cos(u)\cosh(v)\rangle$
8. $\mathbf{r} = \langle \sin(u)\cosh(v), \sin(u)\sinh(v), \cos(u)\rangle$
9. $\mathbf{r} = \langle \sec(u)\sin(v), \sec(u)\cos(v), \tan(u)\rangle$
10. $\mathbf{r} = \left\langle v\sin(u), v\cos(u), v^2\sin(2u)\right\rangle$
11. $\mathbf{r} = \langle e^v\sin(u), e^v\cos(u), e^{-v}\rangle$
12. $\mathbf{r} = \langle \sin(v)\cos(u), \sin(v)\sin(u), \sin^2(v)\cos($

Verify that each of the following parameterizes the unit sphere. Then calculate $\mathbf{r}_u$ *and* $\mathbf{r}_v$*. Is the parameterization orthogonal?*

13. $\mathbf{r} = \langle \cos(u)\sin(v), \cos(v), \sin(u)\sin(v) \rangle$
14. $\mathbf{r} = \langle \sin(v)\sin(u), \cos(v)\sin(u), \cos(u) \rangle$
15. $\mathbf{r} = \langle \sin(v), \cos(u)\cos(v), \sin(u)\cos(v) \rangle$
16. $\mathbf{r} = \langle \cos(u)\cos(v), \sin(u)\cos(v), \sin(v) \rangle$
17. $\mathbf{r} = \left\langle u, \sqrt{1-u^2}\sin(v), \sqrt{1-u^2}\cos(v) \right\rangle$
18. $\mathbf{r} = \left\langle \sqrt{1-u^2}, u\sin(v), u\cos(v) \right\rangle$
19. $\mathbf{r} = \left\langle \frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1} \right\rangle$
20. $\mathbf{r} = \left\langle \frac{2v}{u^2+v^2+1}, \frac{2u}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1} \right\rangle$

Find the surface of revolution obtained by revolving the following surfaces about the x*-axis. Then find* $\mathbf{r}_u$ *and* $\mathbf{r}_v$ *and determine if the parametrization is orthogonal.*

21. $y = x,\ x \text{ in } [0,1]$
26. $y = x+1,\ x \text{ in } [0,1]$
23. $y = x - x^2,\ x \text{ in } [0,1]$
28. $y = x - x^3,\ x \text{ in } [0,1]$
25. $y = \cosh(x),\ x \text{ in } [-1,1]$
30. $y = \sin(x),\ x \text{ in } [0,\pi]$
27. $\mathbf{r}(u,0) = \left\langle u^2, 0, u \right\rangle,\ u \text{ in } [0,1]$
28. $\mathbf{r}(u,0) = \left\langle e^{-u}, 0, u \right\rangle,\ u \text{ in } [0,1]$
29. $\mathbf{r}(u,0) = \langle 2\sin(u), 0, 3+2\cos(u) \rangle$ $u \text{ in } [0,2\pi]$
30. $\mathbf{r}(u,0) = \langle \sin(u), 0, 5+\cos(u) \rangle$ $u \text{ in } [0,2\pi]$

31. Find another parametrization of the sphere of radius R centered at the origin by revolving the upper half circle

$$y = \sqrt{R^2 - x^2}, \quad x \textit{ in } [-R,R]$$

about the x-axis. Is the parametrization orthogonal?

32. Show that every parametric equation of the form

$$\mathbf{r}(u,v) = \langle f(v)\cos(u), f(v)\sin(u), f(v) \rangle$$

is a parametrization of a section the cone $x^2+y^2=z^2$. Is the parametrization orthogonal?

33. A Mobius strip is a surface parametrized by

$$\mathbf{r}(u,v) = \left\langle \cos(u) + v\cos\left(\frac{u}{2}\right)\cos(u), \sin(u) + v\cos\left(\frac{u}{2}\right)\sin(u), v\sin\left(\frac{u}{2}\right) \right\rangle$$

for u in $[0,2\pi]$ and v in $[-0.3, 0.3]$. Graph the Mobius strip with either a graphing calculator or a computer. Also find $\mathbf{r}_u$ and $\mathbf{r}_v$. Is the parameterization of the Mobius strip orthogonal?

34. Show that $\mathbf{r}(t,u,v) = \langle \sin(t)\cos(v), \cos(t)\cos(v), \sin(u)\sin(v), \cos(u)\sin(v) \rangle$ is a parametrization of the sphere in 4 dimensions given by

$$x^2+y^2+z^2+w^2 = 1$$

35. Mercator: Use an identity for the hyperbolic trigonometric functions to prove that

$$\operatorname{sech}^2(A) + \tanh^2(A) = 1$$

Then show that the Mercator parametrization

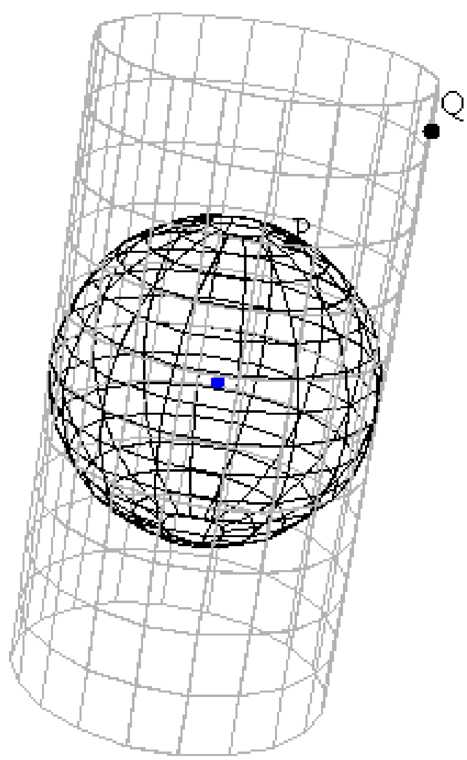
$$\mathbf{r}\left(\theta,\mu\right)=\left\langle R\operatorname{sech}\left(\mu\right)\cos\left(\theta\right),R\operatorname{sech}\left(\mu\right)\sin\left(\theta\right),R\tanh\left(\mu\right)\right\rangle$$

is indeed a parametrization of the sphere of radius R centered at the origin.

36. Mercator: Find $\mathbf{r}_\theta$ and $\mathbf{r}_\mu$ for the Mercator parameterization in exercise 35. Is the parameterization orthogonal? What does that imply about a map constructed as a Mercator projection? Also, what happens to the length of $\mathbf{r}_\mu\left(0,\mu\right)$ as μ approaches ∞? What does this imply about the Mercator projection?

37. Mercator: Show that the Mercator projection in exercise 35 is a surface of revolution (for μ in $(-\infty,\infty)$). How is the half circle in the xz-plane related to the cylinder $x^2+y^2=R^2$ as z approaches ∞ on the cylinder? (More in the Maple worksheet)

38. Cylindrical: The ray from the origin through the point $P(x,y,z)$ on a sphere of radius R intersects a right circular cylinder of radius R at only one point, which we call Q.



If Q is of the form $\left(R\cos\left(u\right),R\sin\left(u\right),Rv\right)$, then what are the coordinates of P in terms of u and v? What is the resulting parameterization of the sphere? Is it the Mercator projection?

39. Find the parametrization of the torus which results from revolving the circle

$$\mathbf{r}(u) = \langle r\sin(u), 0, R + r\cos(u)\rangle$$

for u in $[0, 2\pi]$ about the x-axis, where $R > r > 0$ are constants. *What is a level surface representation of the torus?

41. What is the level surface representation of a sphere of radius R centered at a point (h, k, l)? Explain.

42. Show that every cone whose horizontal cross-sections (i.e., level curves) are circles is of the form

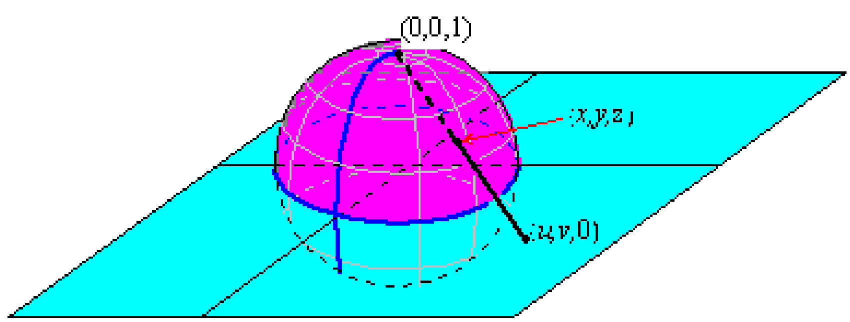
$$x^2 + y^2 = m^2 z^2$$

What is the significance of the m in this equation?

43. Write to Learn: Determine the longitude θ_0 and latitude φ_0 of your present location, and then use (2) to find $\mathbf{r}_\theta$ and $\mathbf{r}_\varphi$ at your present location. In a short essay, present your results and determine the direction (north, south, east, west) that $\mathbf{r}_\theta$ and $\mathbf{r}_\varphi$ are pointing in.

44. Write to Learn: Derive the parameterization of the surface obtained by revolving the curve $y = f(x)$, x in $[a, b]$, about the y-axis. Present and explain your derivation in a short essay.

45. Write to Learn: Stereographic projection assigns to each point $(u, v, 0)$ the point (x, y, z) on the unit sphere that is on the line from the point $(0, 0, 1)$ through the point $(u, v, 0)$.



Use similar right triangles to show that stereographic projection leads to the following parameterization of the sphere:

$$\mathbf{r} = \left\langle \frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1} \right\rangle$$