Geometric Modeling
Based on Space Subdivisions

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Programme

- 1. Introduction
  - Historic.
  - Geometric Modeling $\times$ Solid Modeling.
  - Representation of Curves and Surfaces.
  - Overview.

- 2. Mathematical Background.
  - Mathematical Models for Solids.
  - Complexes.
  - Complete Geometric Complexes(CGCs).

- 3. Representation Schemes.
  - Boundary Representation.
  - Constructive Representation.

Programme (Cont)

- 4. Planar Subdivisions
  - Topological Representations.
  - Adjacency Relationships.
  - The Half-Edge data structure.
  - Euler Operators.

- 5. Spatial Subdivisions
  - Nonmanifold Representation.
  - Adjacency Relationships.
  - The Radial-Edge data structure.
  - Nonmanifold Operators.

- 6. Modeling Process
  - Non-regularized Constructive Geometry.
  - CNRG Combination of CGCs.
  - Heterogeneous Object Modeling.

Historical Facts

- Wireframe Modeling
  - represents objects by edges and points on a surface.
    - produces ambiguous models.

- Surface Modeling (1960)
  - mathematical description of the surfaces bounding the object.
    - few integrity tests for the model.

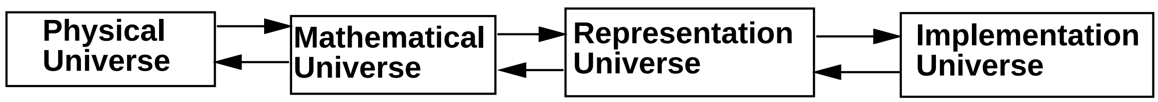
- Solid Modeling (1970)
  - implicitly or explicitly, contain information about the enclosing and connectivity of objects.
    - guarantees the physical realization.
  - CAD-CAM systems used in the industry.

More Recently

- Mixed dimension modeling or non-manifold (1980)
  - objects with internal structure or with dangling elements of lower dimension.
    - solid is bounded by surfaces not (necessarily) locally flat.
  - ACIS (Spatial Technology) - AutoCad

Abstraction Paradigm

- The need of Paradigm



- Paradigm of the Universes
  - Physical *F*.
  - Mathematical *M*.
  - Representation *R*.
  - Implementation *I*.

- Problems
  - Study Phenomena in *F*.
  - Define the Models.
  - Study the Relationships among *R* and *M*.
  - Define Representations for Models in *M*.
  - Study Conversions among Representations.
  - Define Implementation Methods.
  - Compare Strategies in *I*.

Concepts of General Topology

- *Metric Space* is a pair of objects
  - set X
  - real function $d(x, y)$ — metric or distance
    - (positivity)
      - $d(x, y) \geq 0$ and $d(x, x) = 0$, $\forall x, y \in X$.
    - (strict positivity)
      - $d(x, y) = 0 \Leftrightarrow x = y$, $\forall x, y \in X$.
    - (symmetry)
      - $d(x, y) = d(y, x)$, $\forall x, y \in X$.
    - (triangular inequality)
      - $d(x, y) \leq d(x, z) + d(y, z)$, $\forall x, y, z \in X$.

- $(\Re^n,\ d)$ is a metric space
  - $d_2(X, Y) = (\sum_{i=1}^{n} (x_i - y_i)^2)^{1/2}$.
  - $d_\infty(X, Y) = max_i\ (|x_i - y_i|)$.

General Topology (I)

- *Neighborhood* of $x_0 \in (X, d)$ — open ball
  - $B_r(x_0) = \{x \in X : d(x, x_0) < r\},\ r > 0$.

- *Adherence Point* of a set A
  - every neighborhood of a point contains at least one point of A.

- *Closed Set*
  - contains all of its adherence points.

- *Complement* of a set A
  - $\overline{A} = X \backslash A$.

- *Open Set*
  - its complement is closed.

General Topology (II)

- *Closure* of a set A
  - $F(A) = \{$adherence points of $A\}$.
- *Interior* of a set A
  - $I(A) = \{x \in A \mid x \notin F(\overline{A})\}$.
- *Boundary* of a set A
  - $\partial A = \{x \in X \mid x \in F(A)$ and $x \in F(\overline{A})\}$.
- *Regularization* of a set A
  - $R(A) = F(I(A))$.
- *Regular Set*
  - $R(A) = A$.
- *Unity Ball*
  - $S^m = \{x \in \Re^{m+1} \mid < x, x > = 1\}$.

General Topology (III)

- *r-Set*
  - regular and limited set.

- (A, T) and (B, V) are *homeomorphic*
  - there is a function $\Upsilon : (A, T) \to (B, V)$
    - continuous
    - invertible
    - with continuous inverse
  - Υ is a *homeomorphism* of A in B.
  - A and B are topologically *equivalent*.

- *Topological Properties* of a space
  - invariant properties under homeomorphism.

- Homeomorphism plays, in Topology, the role of congruency in Euclidean Geometry.

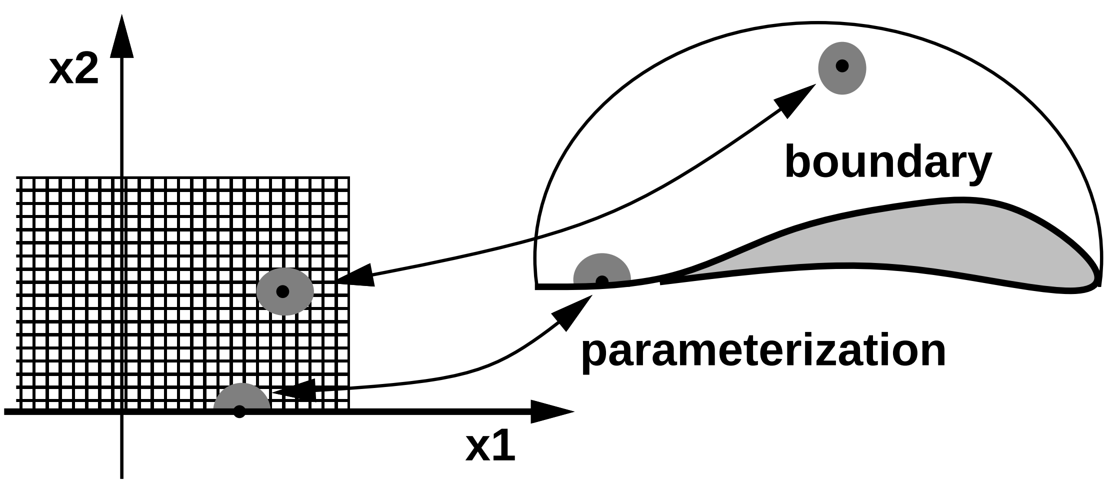
Examples of Homeomorphism

- Composition or inverse of homeomorphism are homeomorphism.
- Invertible Linear Transformations of $\Re^n \to \Re^n$.
- Translations $T_a : \Re^m \to \Re^m,\ T_a(x) = x + a$.
- Two open balls, or two closed balls, or two spheres in $\Re^m$ (scale and translation).
- Every open ball in $\Re^m$ is homeomorphic to the Euclidean space $\Re^m$.
- Stereographic Projection
  - $\varphi : S^m - \{p = (0, ..., 0, 1)\} \to \Re^m$.
    - $\varphi(x) = p + t(x - p) \cap (x_{m+1} = 0),\ t > 0$.
- $\xi : (a,\ cos\ a + 2\pi) \to S^1 - \{(cos\ a,\ sin\ a)\},\ a \in \Re$.
  - $\xi(t) = (cos\ t, sin\ t)$.

Manifolds

- *Manifold without Boundary*
  - topological space M in which every point has a neighborhood homeomorphic to an open set in $\Re^k,\ k > 0$.
    - k is called *dimension* of M.

- *Manifold with Boundary*
  - use $\Re^k_+ = \{(x_1, x_2, ..., x_k) \in \Re^k \mid x_1 \geq 0\}$.
  - a point is on the boundary if for any map of any neighborhood its image is on the boundary of $\Re^k_+$.



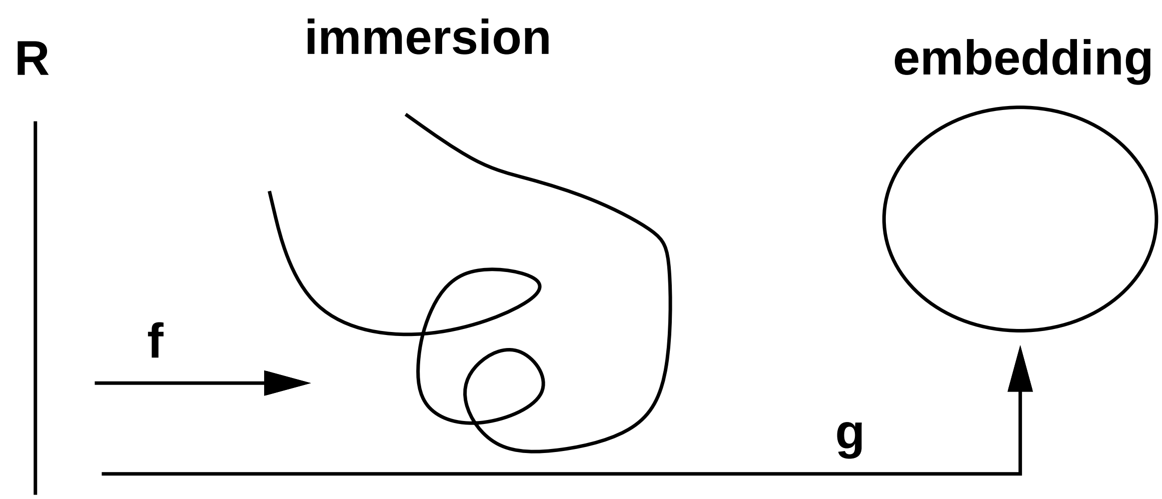
Embedding $\times$ Immersion

- A differentiable map between two manifolds $f : M \to N$ is an *immersion* if

$$\forall p \in M \Rightarrow f'(p) \text{ is } 1-1.$$

  - an immersion is called a parametric surface (curve).
  - an immersed manifold of $\Re^n$ may self-intersect.

- If $f : M \to f(M)$ is also a homeomorphism (with the topology induced by N) then f is an *embedding*.



Objects as Manifolds

- *Surface*
  - is a two-manifold (dimension 2).
- *Realization* of a manifold V in $\Re^3$
  - r-Set A such that ∂A and V are topologically equivalent.
- A manifold is a very general and abstract mathematical concept.
  - it has its own structure and exists independently of any other space.
- When modeling geometrical objects there is a necessity of a more restrictive concept.
  - real objects are part of our world ("live" in $\Re^n$).
  - we need the notion of a manifold inside another.

Submanifold

- A *submanifold* S of a manifold M is a subset $S \subset M$ with:
  - the structure of a manifold.
  - the inclusion map $i : S \to M, i(p) = p$, is an embedding.

- Almost all CG objects are submanifolds of $\Re^2$ or $\Re^3$.
  - *manifold* usually designates an embedded manifold in $\Re^3$.
  - a *two-manifold* is a surface embedded in $\Re^3$.

What is Surface Classification?

- Topological Classification
  - produce a list of standard surfaces.
  - prove that any surface is homeomorphic to a standard surface.
    - homeomorphism is an equivalence class.
    - it is possible to list all classes.

- Geometric Classification
  - in Euclidean Geometry the equivalence is obtained by means of a rigid motion.
  - number of classes is so big that does not make any sense to list.

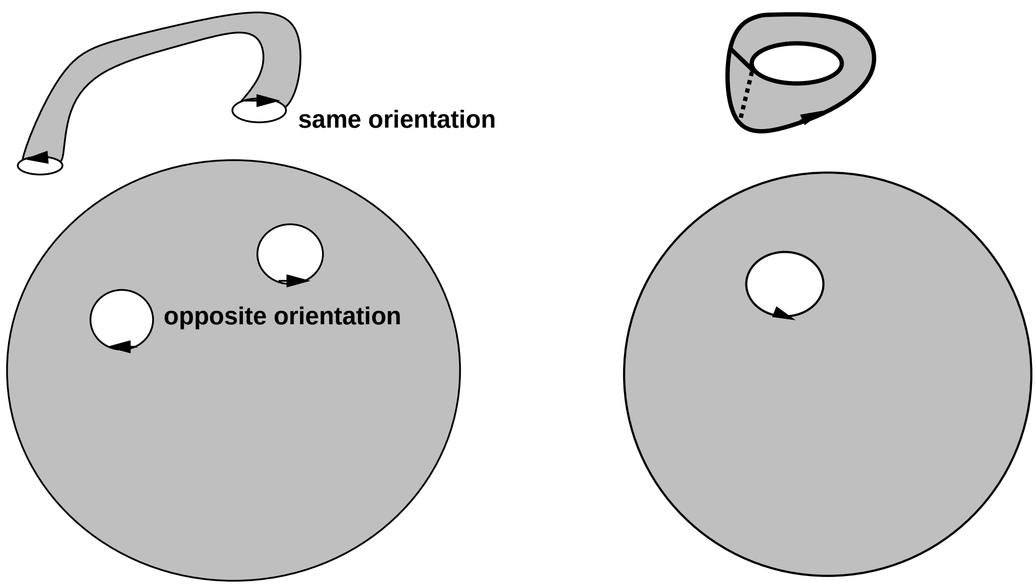
Combinatorial Properties of Manifolds

- Every differentiable manifold is triangulable (Cairns, 1934)
  - possesses a combinatorial structure.
  - can be decomposed in a finite number of vertices, edges and faces.

- Triangulation (decomposition using triangles)
  - (1) any edge is on the boundary of 2 triangles.
  - (2) any vertex is on the boundary of at least 3 triangles forming a cycle around it.

Surface Classification Theorem

- Any triangulable connected surface without boundary is homeomorphic to a standard surface
  - standard orientable surface of genus $G \geq 0$ (sphere with G handles stitched).
  - standard non-orientable surface of genus $G > 0$ (sphere with G Möbius strips stitched).

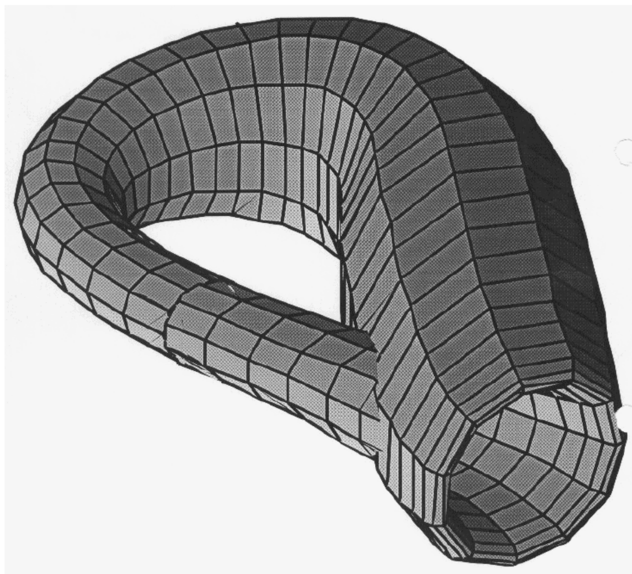


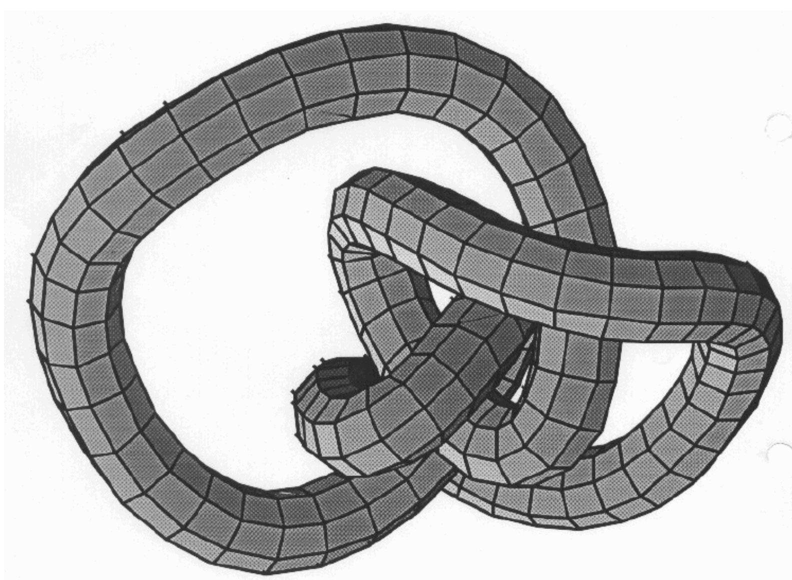
Definition of Orientability

- A surface is *orientable* when it does not contain any Möbius strip.
  - non-orientable surfaces in three dimensions self-intersect.
  - A Klein bottle is made up of two Möbius strips.

- Orientability implies that every triangle of every triangulation can be coherently oriented.
  - a pair of adjacent faces to an edge induce opposite orientations on the edge.
  - it makes sense to think of the interior and exterior of a surface without boundary.
    - exterior points see all triangles oriented counter-clockwise, for instance.

Klein Bottle and Knot surface





Manifold Classification

- A one-manifold without boundary is homeomorphic to an open segment or to a circle.
  - therefore they are orientable.

- For dimensions higher than 3 there are no known classification.

Euler-Poincaré's Characteristic

- Let M be a surface. For any decomposition of M:
  - $\chi(M) = V - A + F$ is constant.
    - spheres: Euler (1752).
    - polyhedra: Poincaré (1890).
    - rigorous prove of invariance for surfaces (1930).

- χ(torus) $= 0$

- χ(sphere) $= 2$

- χ(cylinder) $= 0$

- χ(Möbius strip) $= 0$

- χ(orientable surface genus $G) = 2 - 2G$

- χ(non-orientable surface genus $G) = 2 - G$

Solids

- A *manifold solid* A, with a single shell, is a connected 3-manifold, with boundary and embedded in $\Re^3$.
  - ∂A is a compact orientable surface, without boundary, embedded in $\Re^3$ and with limited variation.
  - the orientation of ∂A makes A the set of interior points of ∂A plus the points of ∂A.

- A *non-manifold solid* is an immersion of several manifold solids.
  - the manifolds can self-intersect in $\Re^3$, but only at dimensions 0 or 1.
  - the objects are topological manifold, but the immersion in $\Re^3$ allows the geometrical coincidence of distinct topological structures.

Space Partition

- The partition of a set U is a collection $U_\alpha, \alpha \in \ I$, of subsets of U such that:
  - $\cup U_\alpha = U$;
  - $U_{\alpha_1} \cap U_{\alpha_2} = \emptyset \; \forall \alpha_1 \neq \alpha_2$.

- The collection of all points of a set define a trivial partition.

Affine Complex

- Restrict the cell geometry.

- Each cell is affine (a convex polytope):
  - $\cap_{i=1}^{m} \{L_i(x) \leq b_i\},\ L_i$: $\Re^n \rightarrow \Re$ affine.

- Generalizes to $\Re^n$ the concept of convex polygons on the plane.

- The boundary of a $K-$dimensional cell has dimension $K - 1$.
  - therefore, affine complexes have cells of all dimensions from 0 to n.

Simplicial Decomposition

- Determine the cell geometry.

- Each cell is a simplex (relatively) open.
  - with $d+1$ points v_i not belonging to the same $d-1$ dimensional hyperplane
    - $\{\sum_{i=0}^{d} x_i v_i; \quad x_i \in (0,1)\}$ is an open simplex of dimension d.
    - dimension 0 are points.
    - dimension 1 are straight segments.
    - dimension 2 are triangles.
    - dimension 3 are tetrahedra.

- Easy of representing, since cells of a certain dimension have boundary with a known number of cells of lower dimension.

- Generates models excessively fragmented.

Simplicial Complex

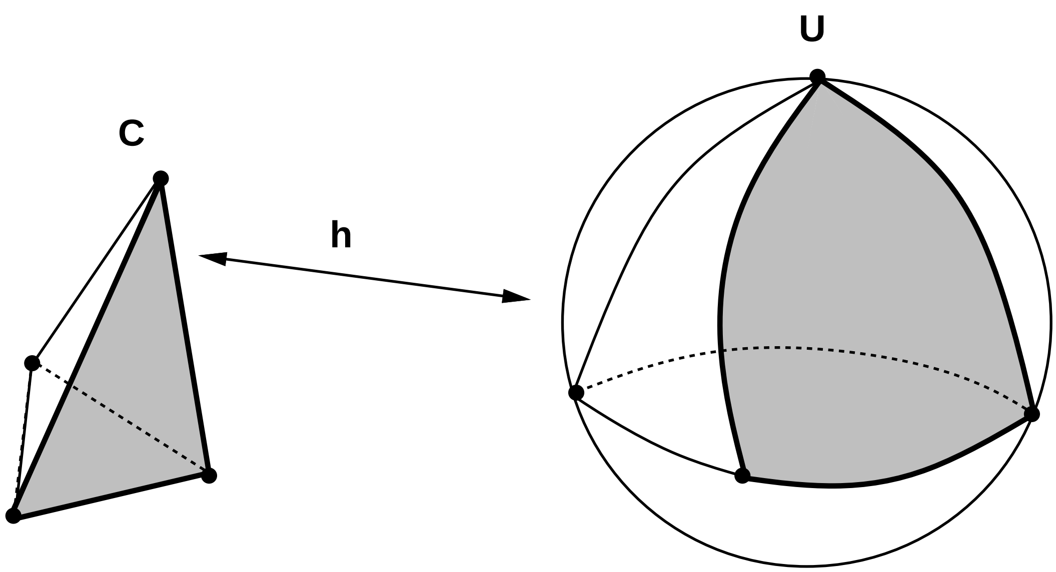
- A simplicial complex T onto a domain $D \subset \Re^n$ is a set of simplices satisfying:
  - $D = \cup_{\sigma \in T}$
  - $\sigma_1 \cap \sigma_2$ is empty or common faces (simplices of lower dimension) of both simplices.
    - oblige that simplex boundaries are complex faces.
  - If D is a compact subset then it intersects a finite number of simplices.

- The cells are closed, but the simplicial complex induces a space partition.

Triangulations

- The *triangulation* of a subset $U \subset \Re^n$ is a homeomorphism $h : C \to U$ of a simplicial complex C onto U.
  - C induces a cellular decomposition in U called a triangulation of U.

- Triangulable sets are called *topological polyhedra*.



Solid Description

- Description for pointsets:
  - by coordinates (parametrically);
  - by density (implicitly).

- Corresponding representations:
  - boundary, B-rep (*Boundary representation*);
  - volumetric, CSG (*Constructive Solid Geometry*).

Boundary Representation

- Solid is defined indirectly, by the surface, compact and without boundary, delimiting it (its boundary).

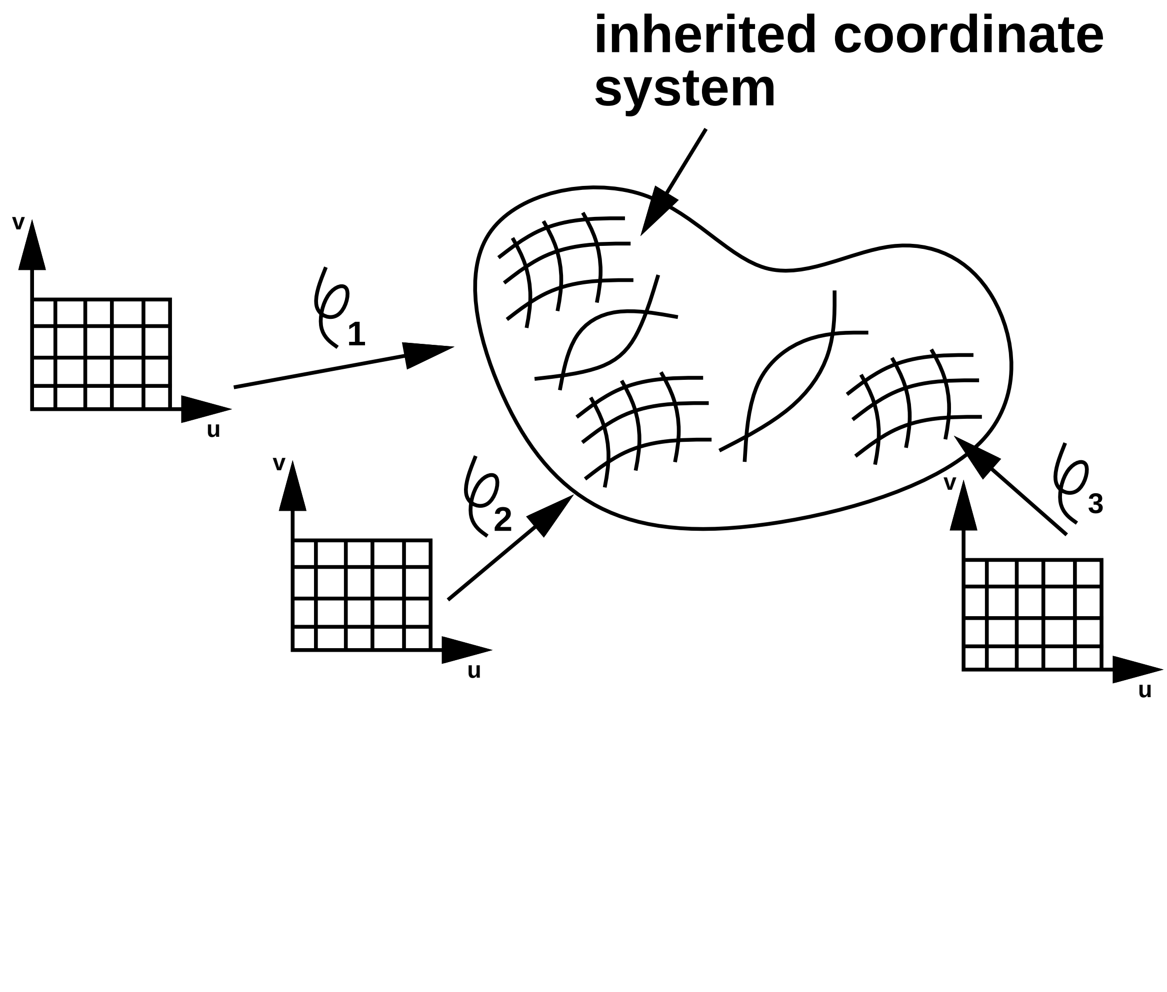
- This surface is described parametrically by a map (parameterization or local chart),
  - $\varphi : U \subset \Re^2 \rightarrow \Re^3$,
  - $\varphi(u, v) = (\varphi_1(u, v),\ \varphi_2(u, v),\ \varphi_3(u, v)) = (x, y, z)$.

- The parameterization establishes a coordinate system onto the surface, inherited from a coordinate system onto the plane.

- It maybe impossible to cover (describe) all of the surface with a single parameterization.

BRep (Cont)

- Piecewise Description
  - several parameterizations define surface patches, which are glued together creating an atlas.
    - may overlap



Boundary Restrictions

- Conditions:
  - the surface does not self-intersect
    - must be an embedded manifold.
  - normal vector, $(\partial\varphi/\partial u \times \partial\varphi/\partial v)$, does not vanish.

- The normal is used to determine the interior and exterior of the solid.
  - e.g., pointing, locally, to the exterior of the solid.

Example

- Unit sphere centered at the origin:
  - $f(\theta, \phi) = (cos\theta\ sin\phi,\ sin\theta\ sin\phi,\ cos\phi)$.
- Normal vector is given by $\frac{\partial f}{\partial \theta} \times \frac{\partial f}{\partial \phi} =$
  - $(-sin\theta\ sin\phi,\ cos\theta\ sin\phi,\ 0) \times$
  - $(cos\theta\ cos\phi,\ sin\theta\ cos\phi,\ -sin\phi)\ =$
  - $-sin\phi(cos\theta\ sin\phi,\ sin\theta\ sin\phi,\ cos\phi)$.
- If $\phi = \pi$ or $\phi = 0$ the normal is not defined at the poles:
  - $U = \{(\theta, \phi) \in \Re^2;\ 0 < \phi < \pi;\ 0 \leq \theta < 2\pi\}$.

Piecewise Linear Representation

- A parameterized surface and geometrically complicate can be approximated by a piecewise linear surface.
- Just polygonize the parameterization domain
  - generally, with triangles or quadrilaterals.
- Each vertex is mapped and linked to adjacent vertices.
  - this process creates a polygonized surface, called a polygonal mesh.
- The geometry is approximated
  - but quite simple.

Polygonal Mesh

- A polygonal mesh is an affine complex.
  - each edge delimits at most two polygons.
  - the intersection of two polygons is an edge, a vertex or empty.

- Basic operations with polygonal meshes:
  - find all edges incident to a vertex;
  - find all polygons sharing and edge or vertex;
  - find and edge delimiting a polygon;
  - exhibit the mesh.

- Implementation for polygonal meshes:
  - explicit;
  - pointers to vertex list;
  - pointers to edge list;
  - *winged-edge*.

Explicit Implementation

- Each polygon explicitly holds a vertex coordinate list:

$$P = \{(x_1, y_1, z_1), (x_2, y_2, z_2), ..., (x_n, y_n, z_n)\}.$$

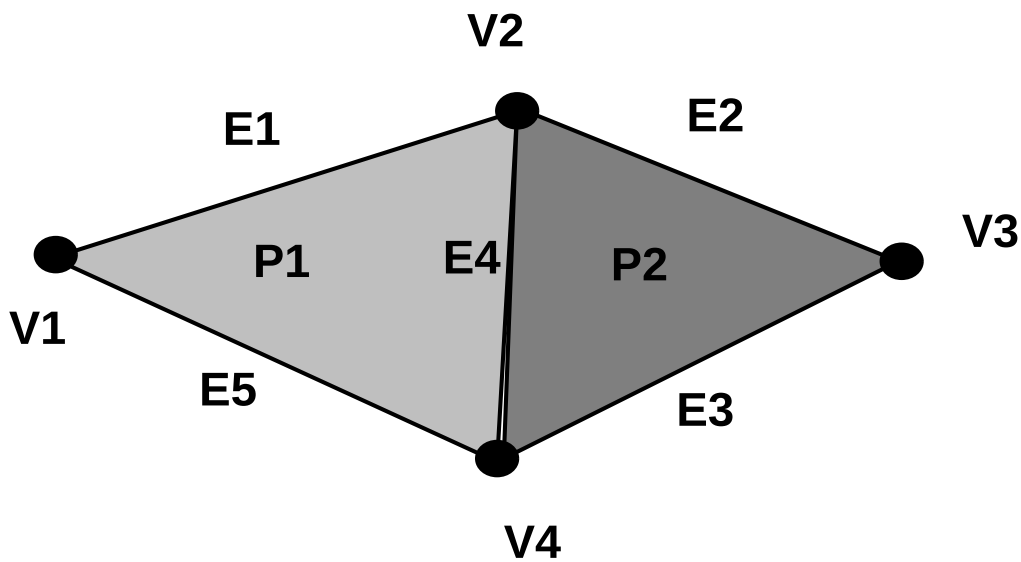
- Information redundancy.
  - duplicate vertices.
  - each edge is drawn twice to exhibit the mesh.
    - causes problems for plotters or films.

- *Queries* are complicated.
  - adjacency relationships are not explicitly stored.
  - forces the execution of geometrical algorithms.

Implementation Using List of Vertices

- Vertices are kept in a list.
- Each polygon references their vertices.
- Less memory spent.
- Vertex coordinates are easily changed.
- Polygons sharing an edge are difficult to find.
- Shared edges are drawn two times.

Simple Example



- $V = \{$
  - $V_1 = (x_1, y_1, z_1)$,
  - $V_2 = (x_2, y_2, z_2)$,
  - $V_3 = (x_3, y_3, z_3)$,
  - $V_4 = (x_4, y_4, z_4)$
- $\}$;
- $P_1 = \{V_1, V_2, V_4\}$;
- $P_2 = \{V_4, V_2, V_3\}$.

Implementation using List of Edges

- There is an edge list.
  - the polygons have references to their edges.
  - $V = \{$
    - $V_1 = (x_1, y_1, z_1)$,
    - $V_2 = (x_2, y_2, z_2)$,
    - $V_3 = (x_3, y_3, z_3)$,
    - $V_4 = (x_4, y_4, z_4)$
  - $\}$;
  - $E_1 = \{V_1, V_2, P_1, \lambda\}$;
  - $E_2 = \{V_2, V_3, P_2, \lambda\}$;
  - $E_3 = \{V_3, V_4, P_2, \lambda\}$;
  - $E_4 = \{V_2, V_4, P_1, P_2\}$;
  - $E_5 = \{V_4, V_1, P_1, \lambda\}$;
  - $P_1 = \{E_1, E_4, E_5\}$;
  - $P_2 = \{E_2, E_3, E_4\}$.

Properties

- The edges are drawn traversing the edge list.
- To find out the edges incident to a vertex requires a geometrical algorithm.
- Two references for the two polygons of the edge.
  - λ, if there is just one.

Winged-Edge

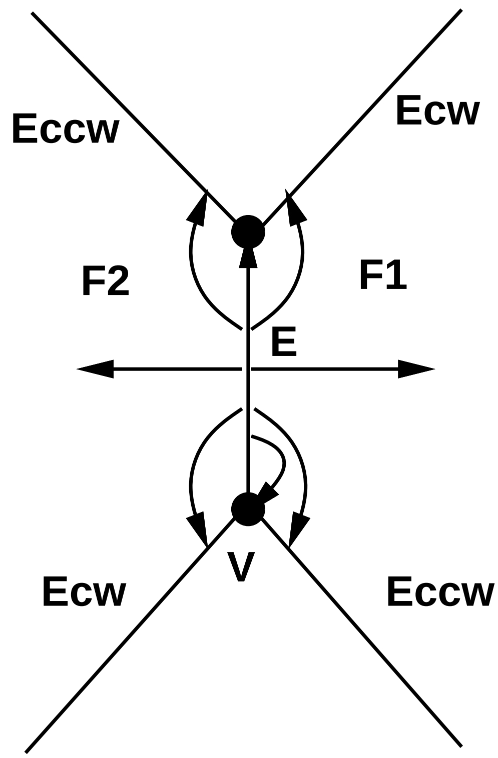
- Created in 1974 by Baungart, it was an accomplishment for BRep data structures.

- Stores information in the structure associated to edges (number of fields is known)
  - an edge is delimited by two vertices and delimits its two polygons, called faces.

- Vertex, edge and face lists are kept.

- Each edge points to:
  - two faces sharing it.
  - to its initial vertex.
  - to four edges succeeding and preceding the ordered cycle of edges around the two faces.

Winged-Edge Diagram



- The nine adjacency relationship types are obtained directly from the data structure.

- The updating of the data structure is difficult
  - made through the use of Euler operators
    - after each model update $V - A + F = 2$ still holds.

Volumetric Representation

- Solid is defined by a set of density values.
  - a table has points with wood density.
  - the complement are points with air density.

- The pointset of $\Re^n$ associated to a density interval forms the pointset defining the solid.

- Describes the object surface, implicitly, by an equation: $F(X) = c;\ X \in \Re^n,\ c \in \Re$.
  - F (implicit function) of class C^k.

- $c \in \Re$ is said a regular value of F if:

$$\forall p \in F^{-1}(c) \Rightarrow \nabla F_p = (\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}) \mid_p \neq 0.$$

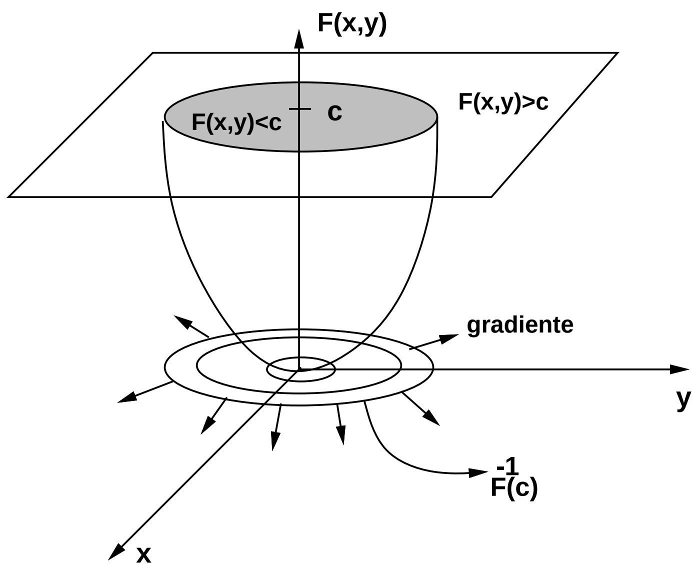
- $n = 2$ or $n = 3$ for curves and implicit surfaces.

Paraboloid

- $F(x, y) = x^2 + y^2$ is a paraboloid in $\Re^3$.
  - $c > 0 = c^{te}$ generates level curves – circles.

- $\nabla F = (2x, 2y)$ vanishes at the origin.
  - 0 is not a regular value of F.
  - $F(x, y) = 0$ does not define an implicit curve.
    - $F^{-1}(0) = (0, 0)$.

- The gradient vector is perpendicular to the level curve.



Spherical Shells

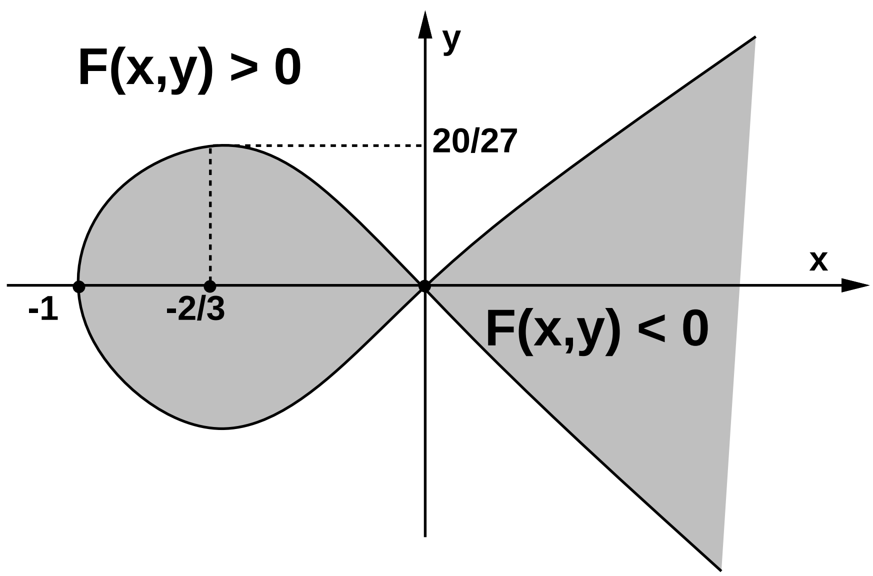
- Defined implicitly by

$$F(x, y, z) = x^2 + y^2 + z^2.$$

- For all $k > 0$, $F^{-1}(k)$ represents the surface of a sphere in $\Re^3$.
  - 0 is not a regular value of F:
    - $F^{-1}(0) = (0, 0, 0)$.
    - $\nabla F = (2x, 2y, 2z)$ vanishes at the origin.

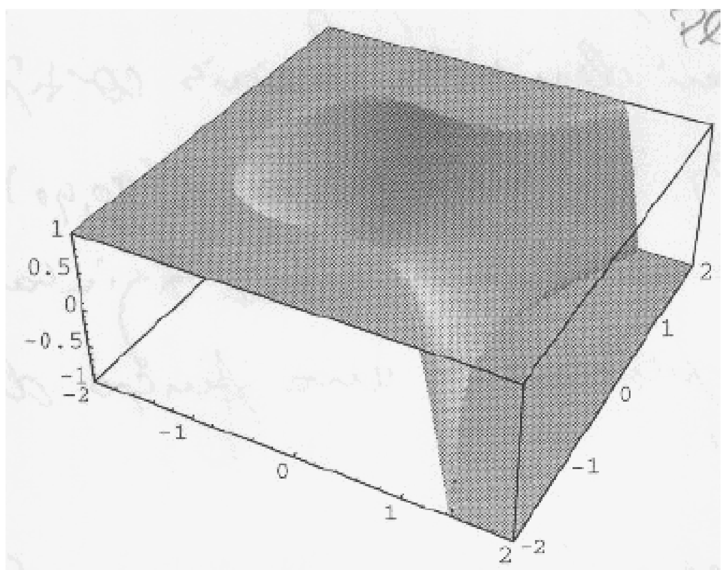
Lace

- Defined implicitly by $F(x, y) = y^2 - x^2 - x^3$.
  - 0 is not a regular value of F:
    - $F^{-1}(0)$ is a curve with a singularity.
    - $\nabla F(x, y) = (2y, -3x^2 - 2x)$ vanishes at the origin.



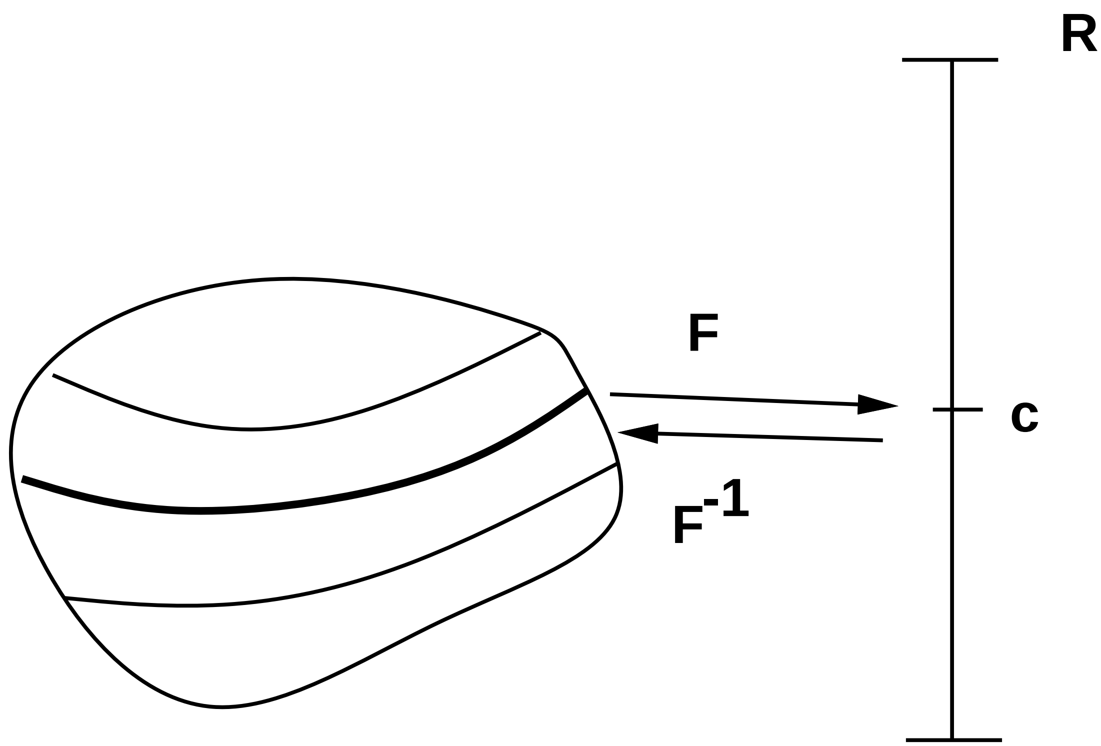
Beyond One Dimension

- Looking at F as a level surface of a function $H : \Re^3 \to \Re$, $H(x,y,z) = -z + y^2 - x^2 - x^3$.
  - 0 is a regular value of H (there is no singularity).
    - $\nabla H(x,y,z) = (-3x^2 - 2x, 2y, -1)$.
    - $\nabla(0,0,0) = (0,0,-1)$.



Implicit Object

- A subset $O \subset \Re^n$ is called an implicit object if there is:
  - $F : U \to \Re,\ O \subset U,$
  - $V \subset \Re \mid O = F^{-1}(V)$ or $O = \{F(p) \in V\}$.



- An implicit object is regular if F satisfies the regularity condition.

Point Classification

- An implicit object is valid if it defines an embedded surface in $\Re^n$.

- If $F(p), p = (x, y, z)$
  - $> 0 \ \Rightarrow p \in$ exterior of O;
  - $= 0 \ \Rightarrow p \in$ boundary of O;
  - $< 0 \ \Rightarrow p \in$ interior of O.

- Function F classifies the points of the space
  - allows to decide whether a point is in the interior, boundary or exterior of the object.

CSG Representation Scheme

- CSG operations are an effective method for building complicate objects out of simple objects.
  - *boolean* regularized pointset operations:
    - $\cup$ - union;
    - $\cap$ - intersection;
    - $\backslash$ - difference.

- A CSG model is represented by a tree.
  - nodes hold set operations or geometrical transformations.
  - leaves hold primitive objects.

- A CSG solid is defined as a pointset of $\Re^n$ satisfying:

$$F(X) \leq 0,\ F : \Re^3 \rightarrow \Re \text{ of class } C^1.$$

Boolean Operations

- The *boolean* operations are defined by:
  - $F_1 \cup F_2 = min\,(F_1, F_2)$;
  - $F_1 \cap F_2 = max\,(F_1, F_2)$;
  - $F_1 \backslash F_2 = F_1 \cap \overline{F_2} = max\,(F_1, -F_2)$.